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COMMENT

The generalized damped cubic equation: integrability and general solution

P G Estévez

Departamento de Fisica Teorica, Facultad de Ciencias, Universidad de Salamanca, 37008 Salamanca, Spain

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Abstract. In this comment the nonlinear ordinary differential equation of the form $u''+f_1u'+f_2u+f_3u^3=0$ representing a generalized damped cubic equation is fully analysed. We find the general expression for f_2 in terms of f_1 and f_3 that renders the equation integrable. Also we show that if f_2 satisfies the above mentioned condition, the former nonlinear differential equation can always be reduced through adequate changes of variable to the second Painlevé transcendent.

There exist a large variety of nonlinear physical systems that are described mathematically through the generalized damped anharmonic oscillator, which is generally given as a nonlinear ordinary differential equation (ODE) of the form:

$$\mathrm{d}^2u(t)/\mathrm{d}t^2+f_1(t)\,\mathrm{d}u(t)/\mathrm{d}t+f_2(t)u(t)+f_3(t)u(t)^3=0. \tag{1}$$

The differential equation (1) has recently been studied in [1] in the case in which $f_2(t)$ and $f_3(t)$ are constants corresponding to the physical situation of a damped kink. In this reference functional forms of $f_1(t)$ are explicitly obtained for the allowed integrable cases. Also the general solution of $u(t)$ for these integrable cases is given. Another particular case of (1) which has also received some attention is the one with $f_2(t)=0$ (a special case of the modified Emden equation) whose integrability has been analysed in [2–4].

More recently a series of papers has appeared [5–7], in which an attempt to identify the full integrability of (1) using the Painlevé test is made for the general case. In [6] and [7] the authors apply the Painlevé test to (1) and in so doing they find the relationship that the set of functions $\{f_1(t),f_2(t),f_3(t)\}$ have to verify if one wishes (1) to be integrable. Such a relationship among the three $\{f_1(t),f_2(t),f_3(t)\}$ takes the following form:

$$\begin{aligned}9f_3^{(4)}f_3^3&-54f_3^{(3)}f_3^{(1)}f_3^2+18f_3^{(3)}f_3^3f_1-36\{f_3^{(2)}\}^2f_3^2+192f_3^{(2)}f_3\{f_3^{(1)}\}^2-78f_3^{(2)}f_3^{(1)}f_3^2f_1\\&+36f_3^{(2)}f_3^3f_2+3f_3^{(2)}f_3^3f_1^2-112\{f_3^{(1)}\}^4+64\{f_3^{(1)}\}^3f_3f_1+6\{f_3^{(1)}\}^2f_1^{(1)}f_3^2\\&-72\{f_3^{(1)}\}^2f_3^2f_2+90f_3^{(1)}f_2^{(1)}f_3^3-27f_3^{(1)}f_1^{(2)}f_3^3-57f_3^{(1)}f_1^{(1)}f_3^3f_1\\&+72f_3^{(1)}f_3^3f_2f_1-14f_3^{(1)}f_3^3f_1^3-54f_2^{(2)}f_3^4-90f_2^{(1)}f_3^4f_1+18f_1^{(3)}f_3^4\\&+54f_1^{(2)}f_3^4f_1+36\{f_1^{(1)}\}^2f_3^4-36f_1^{(1)}f_3^4f_2\\&+60f_1^{(1)}f_3^4f_1^2-36f_3^4f_2f_1^2+8f_3^4f_1^4=0\end{aligned} \tag{2}$$

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where $f_i^{(n)} = \mathrm{d}^n f_i/\mathrm{d}t^n$ and $i = 1, 2, 3$. As can easily be seen, condition (2) looks rather complicated and thus the authors of [6] and [7] restrict themselves to study some particular cases of such a condition. The purpose of this comment is twofold. We will be able to explicitly solve the constraint (2) in such a way that an expression yielding f_2 as a function of f_1 and f_3 will be given. One can then use that expression to check whether a given damped equation (1) is integrable or not by direct substitution of the functions in a much more direct way that using (2).

The second purpose of this comment is to show that for any set of functions $\{f_1(t), f_2(t), f_3(t)\}$ satisfying the integrability condition the ODE (1) can easily be written in the form:

$$\mathrm{d}^2U/\mathrm{d}T^2 - (AT+B)U + U^3 = 0 \tag{3}$$

where A and B are constants. This is the form of the second Painlevé transcendent [9] whose solutions are known to be uniform functions of T and free of movable branch points. *Only in the particular case $A = 0$ the equation (3) can be solved in terms* of elementary elliptic functions. We shall now prove below these announced results.

Let us first give the form of the solved constraint. Let us define the function $M(t)$ in the following form:

$$M(t) = (36f_3^3\lambda(t)^2)^{-1}[6f_3f_3^{(2)} - 7\{f_3^{(1)}\}^2 + 2f_1f_3f_3^{(1)} + 8f_3^2f_1^2 + 12f_3^2f_1^{(1)} - 36f_3^2f_2]. \tag{4}$$

Where $\lambda(t)$ is given by

$$\lambda(t) = f_3^{-1/6}\exp\left((-1/3)\int f_1\,\mathrm{d}t\right). \tag{5}$$

After some elementary manipulations one can show that the condition (2) can be written in terms of $M(t)$ as

$$\mathrm{d}^2M/\mathrm{d}t^2 + (3f_3)^{-1}[f_1f_3 - f_3^{(1)}]\,\mathrm{d}M/\mathrm{d}t = 0. \tag{6}$$

In [7] only the $M(t) = 0$ cases are analysed. Let us now perform the uniform change of variables in (6):

$$\mathrm{d}T = \Phi(t)\,\mathrm{d}t \tag{7a}$$

$$\Phi(t) = f_3^{1/3}\exp\left((-1/3)\int f_1\,\mathrm{d}t\right). \tag{7b}$$

With this change, equation (6) becomes

$$\mathrm{d}^2M/\mathrm{d}T^2 = 0 \tag{8}$$

whose trivial integral is $M(T) = AT + B$ (A and B arbitrary constants). Using now (4), (5), (6) and (7a, b) in this last expression for $M(T)$ we obtain

$$f_2 = (36f_3^2)^{-1}[6f_3f_3^{(2)} - 7\{f_3^{(1)}\}^2 + 2f_1f_3f_3^{(1)} + 12f_3^2f_1^{(1)} + 8f_3^2f_1^2] \\ - \left[B + A\int \mathrm{d}t\,f_3^{1/3}\exp\left((-1/3)\int f_1\,\mathrm{d}t\right)\right]\left[f_3^{2/3}\exp\left((-2/3)\int f_1\,\mathrm{d}t\right)\right] \tag{9}$$

which yields the promised solution of the constraint. It is a trivial matter to show that all the particular cases studied in [6] and [7] satisfy equation (9). For instance, this equation allows us *to check* easily whether equation (1) is integrable or, given f_1 and f_3, *to determine* the functional forms of f_2 *for which the equation is integrable.*

The result follows easily from the above analysis. Let us perform in equation (1) the following change of variables:

$$\mathrm{d}T = \Phi(t)\,\mathrm{d}t \tag{10a}$$

$$u(t) = \lambda(t)U(T) \tag{10b}$$

where $\lambda(t)$ and $\Phi(t)$ are given respectively by (5) and (7*b*). Then equation (1) takes the form:

$$\mathrm{d}^2U/\mathrm{d}T^2 - M(T)U + U^3 = 0. \tag{11}$$

Since, as we have already seen, the equation is only integrable if and only if $M(T) = AT + B$, we finally obtain:

$$\mathrm{d}^2U/\mathrm{d}T^2 - (AT+B)U + U^3 = 0. \tag{12}$$

For $A = 0$ is a trivial nonlinear differential equation whose general solution can be given in terms of elliptic functions. For general A and B this is the second Painlevé transcendent. The general solution can be expressed as a power series in which the coefficients depend upon the initial conditions [9-10].

The example we have developed in this comment can also serve as a first step towards a similar analysis to be performed in the case of partial nonlinear differential equations (PDEs). Several damped nonlinear PDEs are also of interest in a large variety of physical problems. This and other unified formalisms treating related problems will be the subject of a forthcoming report to be published elsewhere.

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